Thermalization of a particle by dissipative collisions

Philippe A. Martin
Institut de Physique Théorique, Ecole Polytechnique Fédérale de Lausanne, CH-1015, Lausanne, Switzerland

Jarosław Piasecki
Institute of Theoretical Physics, University of Warsaw, Hoża 69, 00 681 Warsaw, Poland

(February 1, 2008)

One considers the motion of a test particle in an homogeneous fluid in equilibrium at temperature T, undergoing dissipative collisions with the fluid particles. It is shown that the corresponding linear Boltzmann equation still posseses a stationary Maxwellian velocity distribution, with an effective temperature smaller than T. This effective temperature is explicitly given in terms of the restitution parameter and the masses.

PACS numbers: 51.10.+y, 44.90.+c

The search for stationary states of granular matter has recently been a subject of interest for both experimental and theoretical reasons [1,2]. Granular matter can be modelized by spherical particles that partially dissipate their kinetic energy at collisions. If $(\mathbf{u},\mathbf{v})$ and $(\bar{\mathbf{u}},\bar{\mathbf{v}})$ denote the velocities of two spheres of mass m and M before and after the collision, they are related by

$$m\bar{\mathbf{u}} + M\bar{\mathbf{v}} = m\mathbf{u} + M\mathbf{v} \tag{1}$$

$$\hat{\boldsymbol{\sigma}} \cdot (\bar{\mathbf{v}} - \bar{\mathbf{u}}) = -\alpha\hat{\boldsymbol{\sigma}} \cdot (\mathbf{v} - \mathbf{u}), \quad 0 < \alpha \leq 1 \tag{2}$$

$$\hat{\boldsymbol{\sigma}}^{\perp} \cdot (\bar{\mathbf{v}} - \bar{\mathbf{u}}) = \hat{\boldsymbol{\sigma}}^{\perp} \cdot (\mathbf{v} - \mathbf{u})$$

where $\hat{\boldsymbol{\sigma}}$ is a unit vector normal to the surface of the spheres at the point of impact, and $\hat{\boldsymbol{\sigma}}^{\perp}$ points in the tangent direction: $\hat{\boldsymbol{\sigma}}^{\perp} \cdot \hat{\boldsymbol{\sigma}} = 0$. The first relation is the conservation of the center of mass momentum, whereas the second one says that the normal component of the relative velocity reverses its direction with a magnitude reduced by the factor α, the so called restitution parameter (the tangent component remains unchanged). Solving (1), (2) for $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ gives

$$\bar{\mathbf{u}} = \mathbf{u} + (1-\mu)(1+\alpha)(\hat{\boldsymbol{\sigma}} \cdot (\mathbf{v} - \mathbf{u}))\hat{\boldsymbol{\sigma}}, \quad \bar{\mathbf{v}} = \mathbf{v} - \mu(1+\alpha)(\hat{\boldsymbol{\sigma}} \cdot (\mathbf{v} - \mathbf{u}))\hat{\boldsymbol{\sigma}} \tag{3}$$

with $\mu = m/(m+M)$. The inverse relation is obtained by exchanging the roles of initial and final velocities and α into α^{-1} in (3).

We consider now the motion of a single particle of mass M (the test particle) in a fluid of particles of mass m in thermal equilibrium at temperature T by means

of the linearized Boltzmann equation. All particles are spheres of diameter d and the test particle undergoes inelastic collisions characterized by the restitution parameter α with the particles of the host fluid. Then, in the homogeneous situation, the probability density $f(\mathbf{v},t)$ for finding the test particle at time t with velocity $\mathbf{v}$ obeys the kinetic equation

$$\frac{\partial}{\partial t} f(\mathbf{v},t) = \rho d^2 \int d\mathbf{u} \int d\hat{\boldsymbol{\sigma}} (\hat{\boldsymbol{\sigma}}\cdot(\mathbf{v}-\mathbf{u}))\theta(\hat{\boldsymbol{\sigma}}\cdot(\mathbf{v}-\mathbf{u})) \times$$

$$\left[\alpha^{-2} f\left(\mathbf{v}-\mu\left(1+\alpha^{-1}\right)(\hat{\boldsymbol{\sigma}}\cdot(\mathbf{v}-\mathbf{u}))\hat{\boldsymbol{\sigma}},t\right)\phi_T\left(\mathbf{u}+(1-\mu)\left(1+\alpha^{-1}\right)(\hat{\boldsymbol{\sigma}}\cdot(\mathbf{v}-\mathbf{u}))\hat{\boldsymbol{\sigma}}\right)\right.$$
$$\left. - f(\mathbf{v},t)\phi_T(\mathbf{u})\right] \tag{4}$$

Here $\rho\phi_T(\mathbf{u})$ is the equilibrium state the fluid having uniform density ρ and Mawellian velocity distribution

$$\phi_T(\mathbf{u}) = \left(\frac{\beta m}{2\pi}\right)^{3/2} \exp\left(-\beta\frac{mu^2}{2}\right), \quad u = |\mathbf{u}|, \quad \beta = \frac{1}{k_B T} \tag{5}$$

In (4), $\mathbf{u}$ is the velocity of a particle in the fluid, $\mathbf{v}$ is the velocity of the test particle and the $\hat{\boldsymbol{\sigma}}$-integral runs on the unit sphere. The velocity arguments of f and ϕ_T in the gain part of the collision term in (4) are precisely those that the particles must have before the collision to produce post-collisional velocities $\mathbf{u}$ and $\mathbf{v}$, according to (3). The term$(\hat{\boldsymbol{\sigma}}\cdot(\mathbf{v}-\mathbf{u}))\theta(\hat{\boldsymbol{\sigma}}\cdot(\mathbf{v}-\mathbf{u}))$ (with $\theta(y)$ the Haeviside function) reflects the fact that the collision frequency is proportional to the velocity of approach of the colliding pair (free motion between encounters). The factor α^{-2} compensates for the contractive transformation (2) in velocity space and ensures that the normalization of f is properly conserved in the course of the time.

The point of this letter is to show that, despite of its dissipative collisions with the particles of the bath, the distribution of test particle still posseses a stationary *Maxwellian* velocity distribution, with a reduced effective temperature T^{eff} that will be explicitly given in terms of the restitution parameter and the masses.

We look for a stationary state setting simply $f(\mathbf{v},t) = f(\mathbf{v})$ and equating the r.h.s. of (4) to zero. With the change of variable $\mathbf{w} = \mathbf{v} - \mathbf{u}$, f has to satisfy

$$0 = \int d\hat{\boldsymbol{\sigma}} \int d\mathbf{w} (\hat{\boldsymbol{\sigma}}\cdot\mathbf{w})\theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{w}) \times$$

$$\left[\alpha^{-2} f\left(\mathbf{v}-\mu\left(1+\alpha^{-1}\right)(\hat{\boldsymbol{\sigma}}\cdot\mathbf{w})\hat{\boldsymbol{\sigma}}\right)\phi_T\left(\mathbf{v}-\mathbf{w}+(1-\mu)\left(1+\alpha^{-1}\right)(\hat{\boldsymbol{\sigma}}\cdot\mathbf{w})\hat{\boldsymbol{\sigma}}\right)\right.$$
$$\left. - f(\mathbf{v})\phi_T(\mathbf{v}-\mathbf{w})\right] \tag{6}$$

Let us perform the $\mathbf{w}-$integration at fixed $\hat{\boldsymbol{\sigma}}$ in a frame

$$\mathbf{w} = w_1\hat{\boldsymbol{\sigma}}_1 + w_2\hat{\boldsymbol{\sigma}}_2 + w_3\hat{\boldsymbol{\sigma}}_3 \tag{7}$$

where $\{\hat{\boldsymbol{\sigma}}_1 = \hat{\boldsymbol{\sigma}},\ \hat{\boldsymbol{\sigma}}_2,\ \hat{\boldsymbol{\sigma}}_3\}$ is an orthonormal system of unit vectors with $\hat{\boldsymbol{\sigma}}_2,\ \hat{\boldsymbol{\sigma}}_3$ in the plane perpendicular to $\hat{\boldsymbol{\sigma}}$. Note that the argument of f does not depend on the variables w_2 and w_3, so the corresponding integrals can be carried out directely on ϕ_T. In terms of the variables (7) we have

$$\begin{aligned}
&\phi_T\left(\mathbf{v} - \mathbf{w} + (1-\mu)\left(1+\alpha^{-1}\right)(\hat{\boldsymbol{\sigma}}\cdot\mathbf{w})\hat{\boldsymbol{\sigma}}\right) \\
&= \left(\frac{\beta m}{2\pi}\right)^{3/2} \exp\left\{-\frac{\beta m}{2}\left(\hat{\boldsymbol{\sigma}}\cdot\mathbf{v} + \left(\alpha^{-1} - \mu\left(1+\alpha^{-1}\right)\right)w_1\right)^2\right\} \\
&\times \exp\left(-\frac{\beta m}{2}(\hat{\boldsymbol{\sigma}}_2\cdot\mathbf{v} - w_2)^2\right)\exp\left(-\frac{\beta m}{2}(\hat{\boldsymbol{\sigma}}_3\cdot\mathbf{v} - w_3)^2\right)
\end{aligned} \tag{8}$$

Hence the integrations on w_2 and w_3 can be readily performed here (as well in $\phi_T(\mathbf{v}-\mathbf{w})$) and the stationary equation takes eventually the form

$$\begin{aligned}
0 = \int d\hat{\boldsymbol{\sigma}} \int_0^\infty dyy\, \Bigg[& f(\mathbf{v} - \eta y\hat{\boldsymbol{\sigma}})\exp\left(-\frac{\beta m}{2}(\hat{\boldsymbol{\sigma}}\cdot\mathbf{v} + (1-\eta)y)^2\right) \\
& - f(\mathbf{v})\exp\left(-\frac{\beta m}{2}(\hat{\boldsymbol{\sigma}}\cdot\mathbf{v} - y)^2\right)\Bigg]
\end{aligned} \tag{9}$$

To obtain (9) we also put $w_1 = \alpha y$ in the first part of the integrand in (6) (gain term) and we defined the new parameter $\eta = \mu(\alpha+1)$.

Now, let us set

$$f(\mathbf{v}) = \exp\left(-\beta M v^2/2\gamma\right),\ v = |\mathbf{v}|$$

omitting the normalization factor. One notices then that the following equality holds

$$M\gamma^{-1}|\mathbf{v} - \eta y\hat{\boldsymbol{\sigma}}|^2 + m(\hat{\boldsymbol{\sigma}}\cdot\mathbf{v} + (1-\eta)y)^2 = M\gamma^{-1}v^2 + m(\hat{\boldsymbol{\sigma}}\cdot\mathbf{v} - y)^2 \tag{10}$$

provided that γ^{-1} has the value (independent of $\mathbf{v}$, $\hat{\boldsymbol{\sigma}}$ and y)

$$\gamma^{-1} = \frac{m(2-\eta)}{M\eta} \tag{11}$$

The relation (10) implies vanishing of the integrand in (9). Introducing the effective temperature T^{eff} by

$$T^{\text{eff}} = \gamma T \tag{12}$$

we conclude that the Maxwellian $f(\mathbf{v}) = \phi_{T^{\text{eff}}}(\mathbf{v})$ is a stationary distribution for our test particle. The explicit formula for the factor γ reducing the temperature in terms of the restitution parameter and the masses reads

$$\gamma = \frac{\alpha+1}{2 + (1-\alpha)m/M} \leq 1 \tag{13}$$

Obviously $\gamma = 1$ when $\alpha = 1$, as it should. For a fixed mass ratio m/M, γ decreases as α is varied from 1 to 0. Thus it takes its smallest possible value $\gamma = (2 + m/M)^{-1} < 1/2$ at $\alpha = 0$ when the collisions are fully dissipative.

This result deserves several comments. Usually the dependence of the stationary state on the kinematical variables is determined by the conservation laws. The stationary Maxwellian velocity distribution for the hard sphere Boltzmann equation follows from the conservation of the total kinetic energy $\frac{1}{2}mu^2 + \frac{1}{2}Mv^2$ at collisions. It is therefore quite remarquable (and not obvious beforehand) that the distribution remains Gaussian when collisions dissipate the kinetic energy. Notice that the integrand in (9) vanishes pointwise, that is, $\phi_{T^{\rm eff}}(\mathbf{v})$ is also stationnary in situations where the collision frequency, proportional to $(\hat{\boldsymbol{\sigma}} \cdot \mathbf{w})\theta(\hat{\boldsymbol{\sigma}} \cdot \mathbf{w})$, is replaced by a more general positive function of $\hat{\boldsymbol{\sigma}} \cdot \mathbf{w}$. A particular case is the so called Maxwell gas for which the collision frequency is independent of the velocity. Moreover, the result holds in any space dimension $d = 1, 2, 3$ with the same formulae (12), (13) for the effective temperature.

These findings must be contrasted to other physical situations where the stationary distribution is definitely not Maxwellian. One example is the Lorentz model with dissipative collisions: a particle, acted upon an uniform external field, undergoes inelastic collisions with randomly distributed static (infinitely heavy) scatterers. Here the dissipation (however weak it may be) sufficies to balance the energy flow from the external field and to guarantee the existence of a stationary state. In the weakly dissipative regime, the stationary velocity distribution is found to behave as $\sim \exp(-(1-\alpha)v^4/2)$ [3,4]. The physics of the Lorentz gas is not the same since there is no thermalization mechanism available for the moving particle. In [5], one considers a one-dimensional system of infinitely many inelastic particles in contact with a thermal bath, modelized by a Fokker-Plank equation in the limit of weak dissipation. In this case, the stationary velocity distribution behaves as $\sim \exp(-Cv^3)$, but here dissipation occurs internally within the many particle system, and not in relation with its interaction with the thermal bath.

Further investigations of interest are the study of the approach to the steady state in the course of time and the possible generalization of our result to a system of several interacting test particles.

Acknowledgements

J.Piasecki acknowledges the hospitality at the Institute of Theoretical Physics of the Ecole Polytechnique Fédérale de Lausanne (Switzerland), and financial support by the KBN (Committee for scientific Research, Poland), grant 2 P03B 03512.

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